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On Finite Groups with a Certain Sylow Normalizer. II

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From the main theorem of [1] on non-Abelian Sylow subgroups, we argue via Schur multipliers to a corollary about Abelian Sylow subgroups. This result was suggested by Professor Graham Higman, and we are grateful to Professor George Glauberman for indicating the relevant results on local control of cohomology.

THEOREM. *Let G be a finite group, P a Sylow p-subgroup of G, p odd. Suppose*: (1) *P is Abelian and*

(2) $|N(P)/C(P)| = 2$.

Then (i) *If G is perfect, then P is cyclic.*

(ii) *If P is noncyclic, then $O^p(G) < G$ or G is p-solvable of p-length* 1.

Proof. It suffices to prove (ii) to obtain (i); in case $O^p(G) = G$, we get $G = O_{p'}(G) \cdot N(P)$, and G has a subgroup of index 2.

(a) G is certainly p-normal, with $P = Z(P)$, so by a theorem of Swan [2, p. 346] the p-parts of the Schur multipliers of G and of $N(P)$ are isomorphic.

(b) Since P is not cyclic, the multiplier of $N(P)$ has nontrivial p-part: If $P = \langle x_1 \rangle \times \langle x_2 \rangle \times A$, we can construct an extension as follows: Let $B = \langle y_1, y_2 \rangle$ where $|y_i| = |x_i|$, $i = 1, 2$; $[y_1, y_2] = z \in Z(B)$, $|z| = p$. Let $Q = B \times A$. If $N(P) = \langle t, P \times C \rangle$ with $C = O_{p'}(C(P))$ and t inverting P, define $M = \langle t, Q \times C \rangle$ with t inverting $Q/\langle z \rangle$ and centralizing z. Then $M/\langle z \rangle \simeq N(P)$ as desired.

(c) Thus, the p-part of the multiplier of G must be nontrivial. So we can find H with $z \in H' \cap Z(H)$ of order p, $H/\langle z \rangle \simeq G$. Let Q be a Sylow p-subgroup of H so that $Q/\langle z \rangle \simeq P$. Then $N_H(Q)/\langle z \rangle \simeq N(P)$, and

$$|N_H(Q)/Q \cdot O_{p'}(N_H(Q))| = 2.$$

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Since H does not split over $\langle z \rangle$, by the theorem of Gaschütz [3, p. 245], neither does Q. Thus, $Q' = \langle z \rangle$ is of order p. Finally, $O^p(G) = G$ forces $O^p(H) = H$, so that hypotheses (1), (2), and (3) of the main theorem of [1] are satisfied. Thus H, and hence G, is p-solvable of p-length 1.

This completes the proof.

References

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3. M. Hall, Jr., "The Theory of Groups," Macmillan, New York, 1959.